ECE317 : Feedback and Control

Lecture :
Steady-state error

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Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- ✓ Block Diagram
- ✓ Linearization
- ✓ Models for systems
  - ✓ electrical
  - ✓ mechanical
  - ✓ example system

Analysis

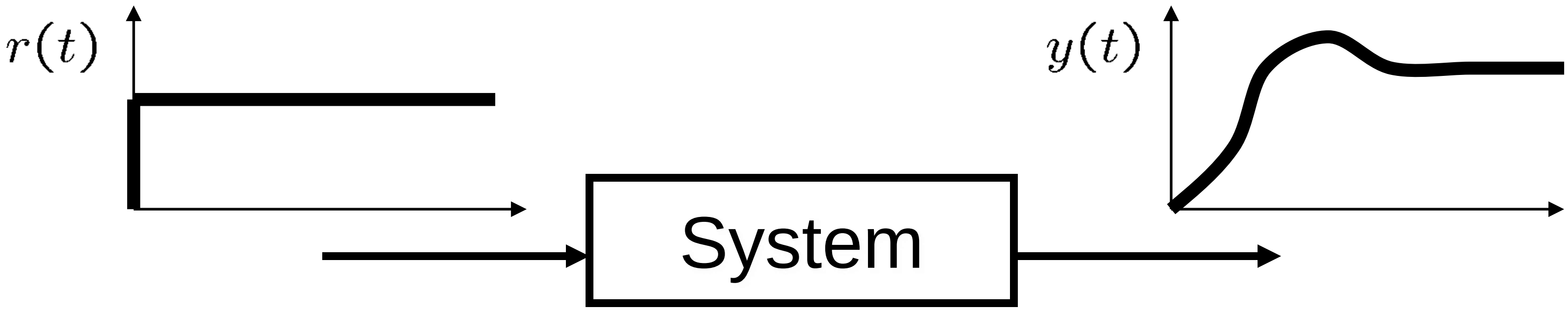
- ✓ Stability
  - ✓ Pole locations
  - ✓ Routh-Hurwitz
- ✓ Time response
  - ✓ Transient
  - Steady state (error)
- ✓ Frequency response
  - ✓ Bode plot

Design

- ✓ Design specs
- ✓ Frequency domain
- ✓ Bode plot
- ✓ Compensation
- ✓ Design examples

Matlab & PECS simulations & laboratories

Time response



- We would like to analyze (stable) system's property by applying a *test input* *r(t)* and observing a time response *y(t)*.
- Time response is divided as

$$y(t) = y_t(t) + y_{ss}(t)$$

Transient (natural) response

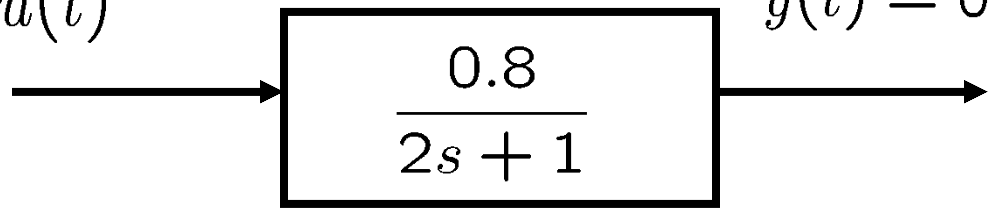
$$\lim_{t \to \infty} y_t(t) = 0$$

Steady-state (forced) response

(after y_t dies out)

Ex: Transient & steady-state responses

$$r(t) = u(t)$$



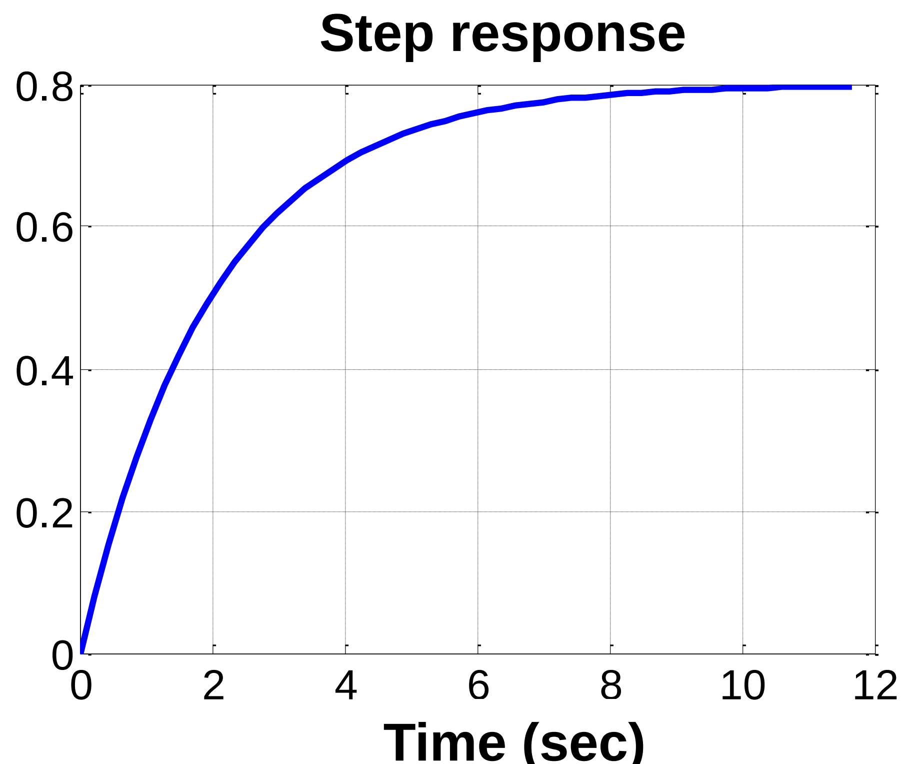
$$y(t) = 0.8 - 0.8e^{-\frac{t}{2}}$$

- Transient response

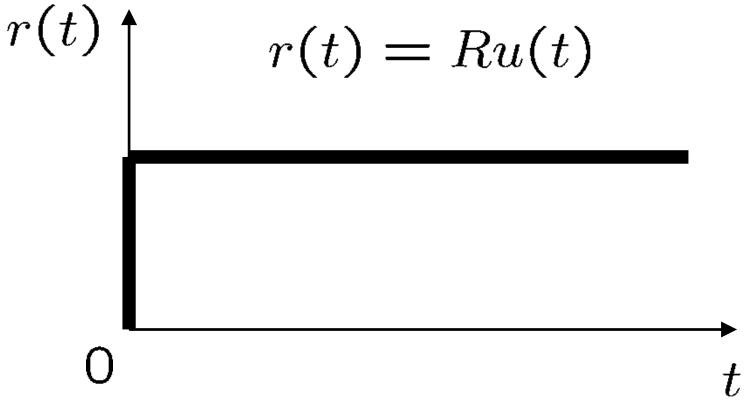
$$y_t(t) = -0.8e^{-\frac{t}{2}}$$

- Steady-state resp.

$$y_{ss}(t) = 0.8$$

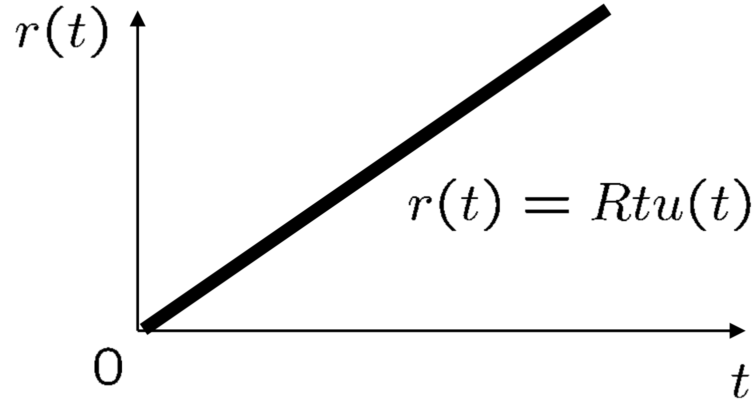


Typical test inputs



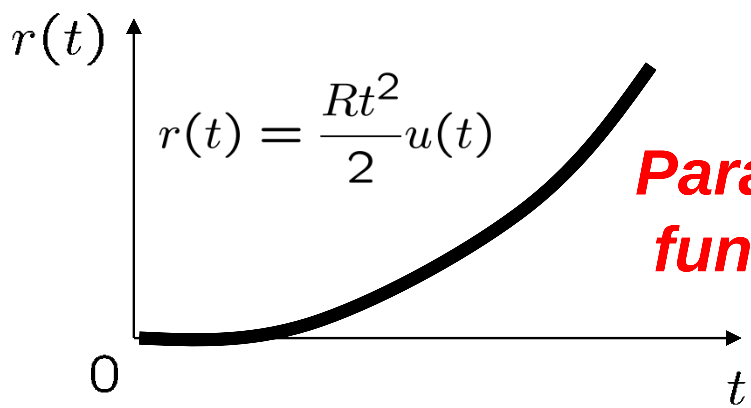
Step function

(Most popular)



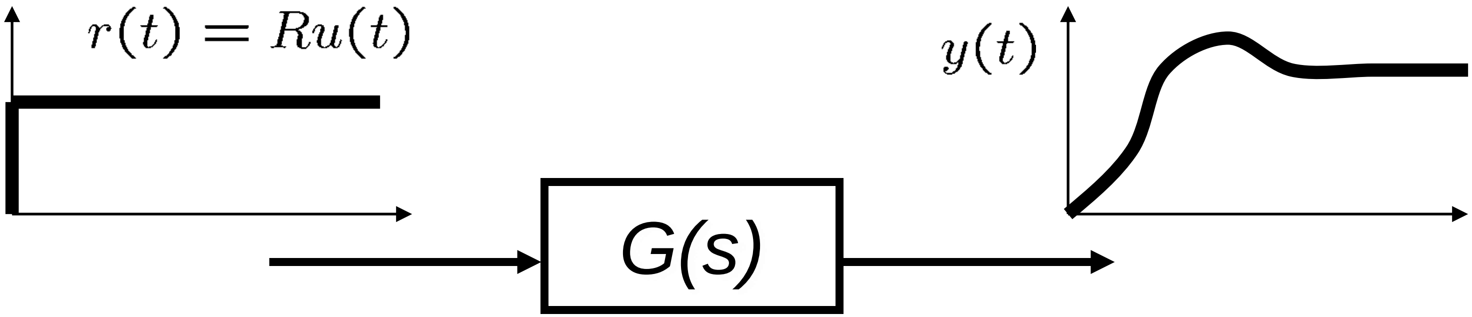
Ramp function

Sinusoidal input
was dealt with earlier
➔ freq. response



Parabolic function

Steady-state value for step input

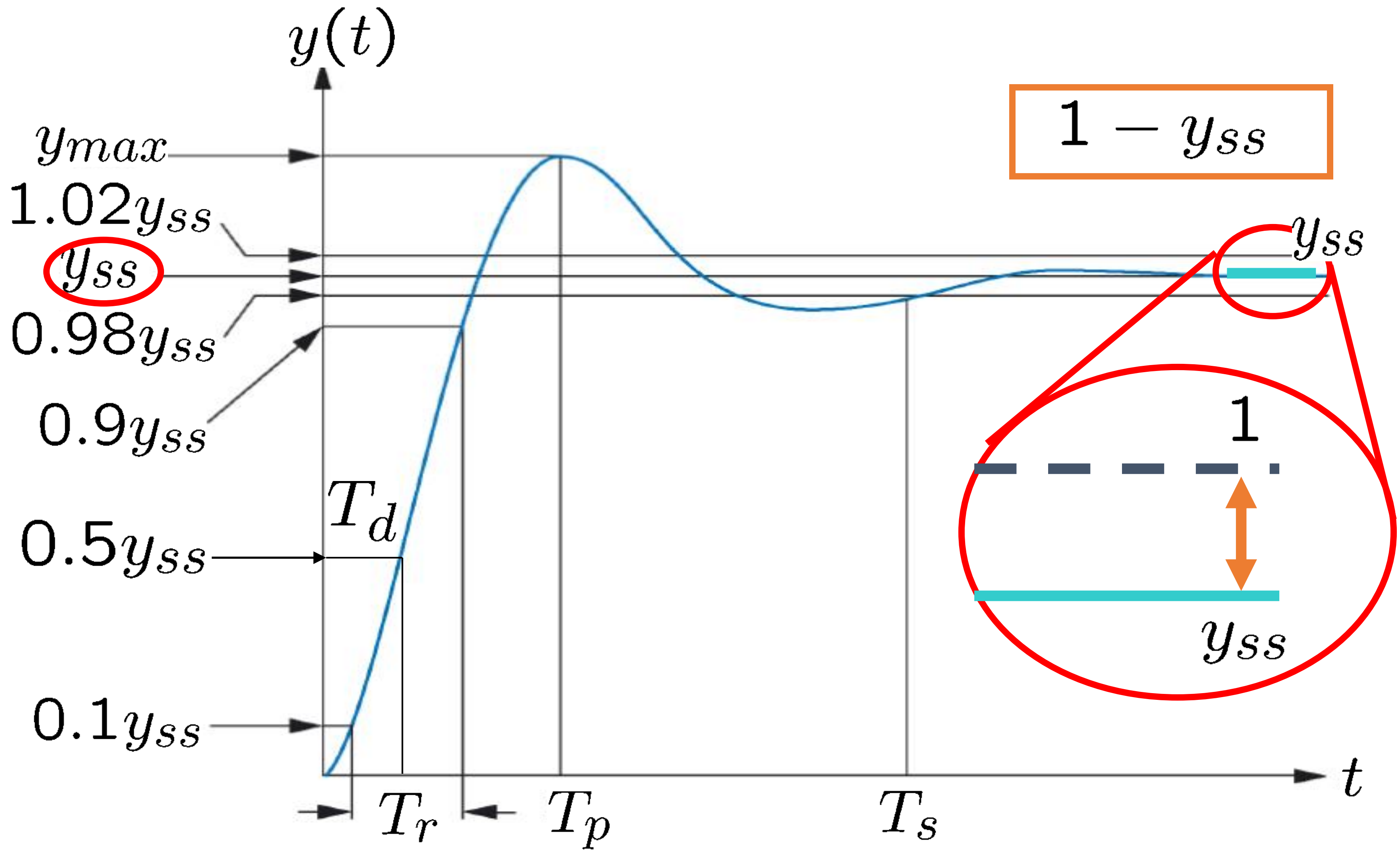


- Suppose that G(s) is stable.
- By the final value theorem:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sG(s)\frac{R}{s} = RG(0)$$

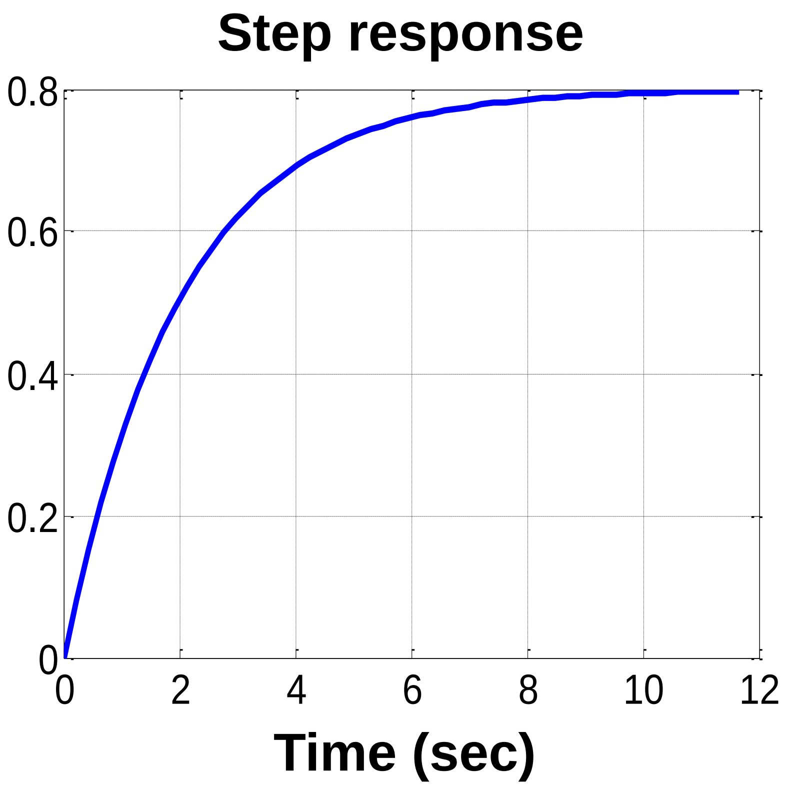
- Step response converges to some finite value, called *steady-state value* y_{ss}

Steady-state error for input *u(t)*



Example revisited

- For the example on Slide 4:
  - Steady-state error : 1-0.8=0.2



Performance measures

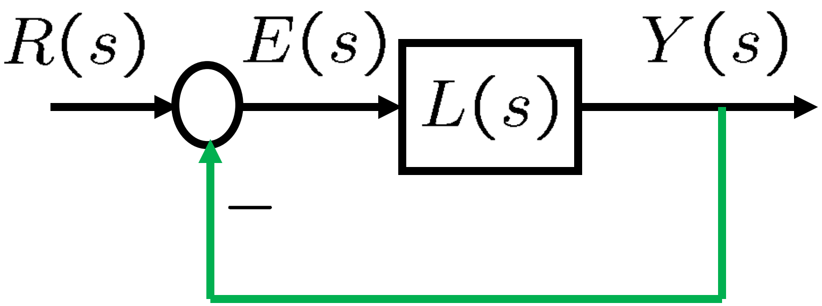
- Transient response
  - Peak value
  - Peak time
  - Percent overshoot
  - Delay time
  - Rise time
  - Settling time
- Steady state response
  - Steady state error

(Previous lectures)

Next, we will connect these measures with s-domain.

(Today's lecture)

Steady-state error of feedback system



Assumptions

- *L(s)* = Plant(s)*Controller(s)
- Unity feedback (no block on feedback path)
- CL system is stable

- Suppose that we want output *y(t)* to track *r(t)*.
- Error $e(t) := r(t) - y(t)$
- Steady-state error

$$e_{ss} := \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{1}{1 + L(s)} R(s)$$

Final value theorem

(Suppose CL system is stable!!!)

Error constants

- CL system's ability to reduce steady-state error e_{ss}
  - "Large error constant" means "large ability".
- Three error constants
  - Step-error (position-error) constant

$$K_p := \lim_{s \to 0} L(s)$$

  - Ramp-error (velocity-error) constant

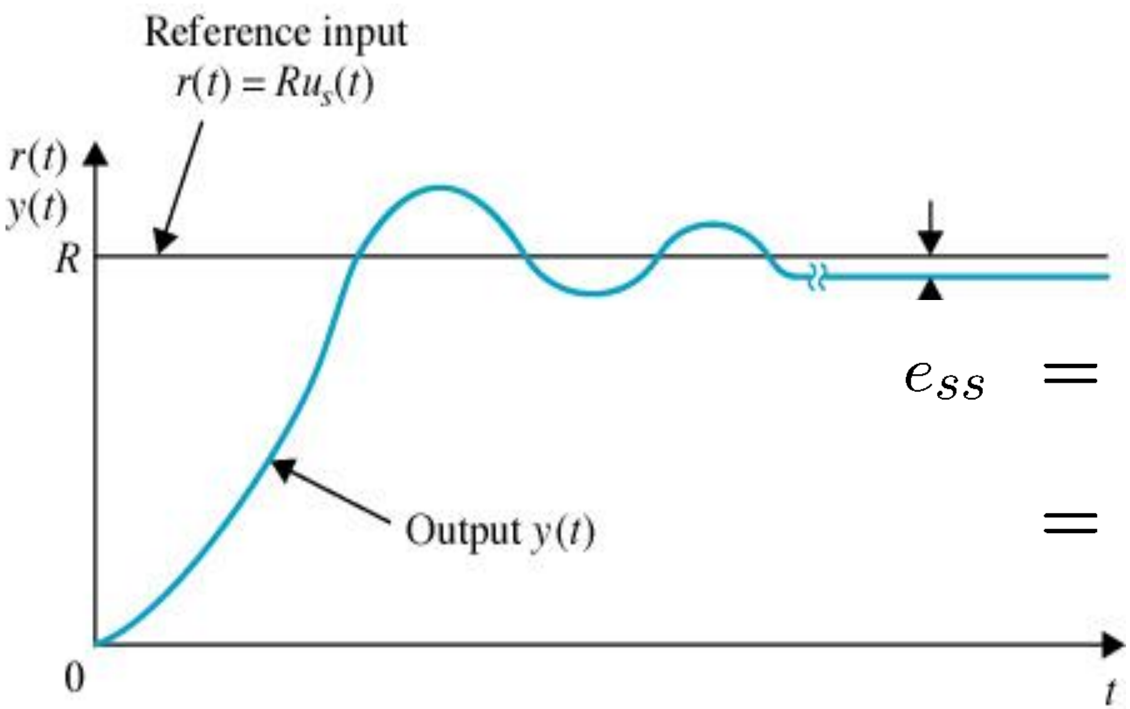
$$K_v := \lim_{s \to 0} sL(s)$$

  - Parabolic-error (acceleration-error) constant

$$K_a := \lim_{s \to 0} s^2 L(s)$$

Steady-state error for step *r(t)*

$$r(t) = Ru(t) \Rightarrow e_{ss} = \frac{R}{1+K_p}$$

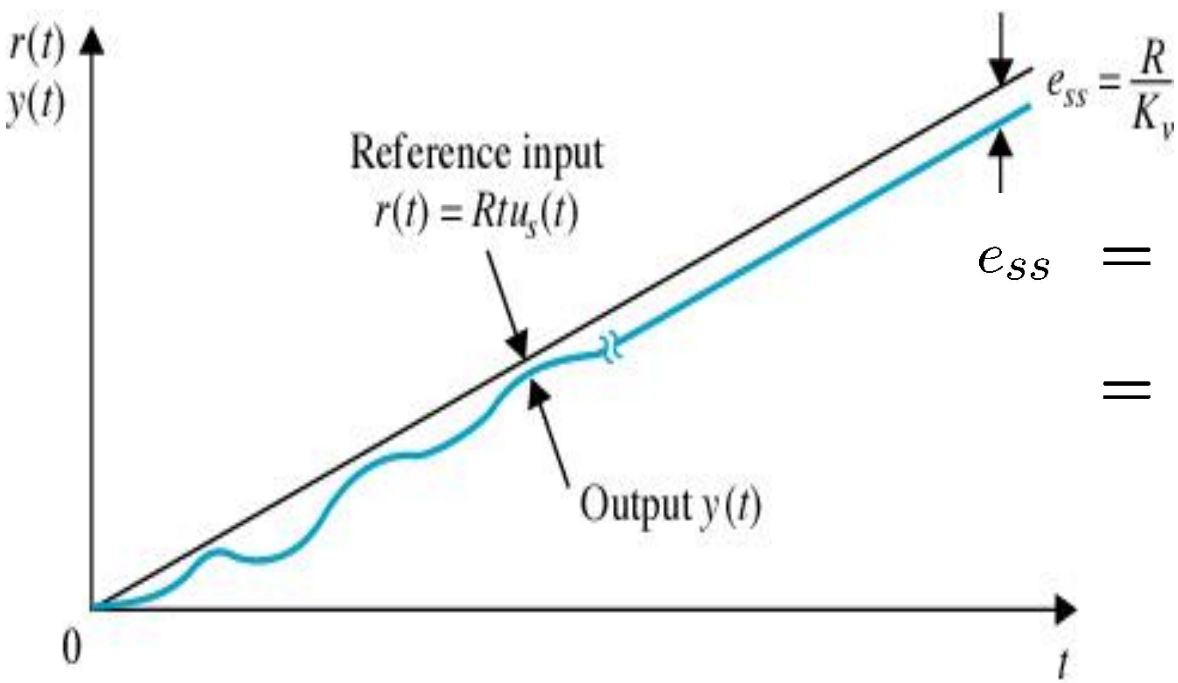


$$e_{ss} = \lim_{s \to 0} s \frac{1}{1+L(s)} \cdot \frac{R}{s}$$

$$= \frac{R}{1+\underbrace{L(0)}_{Kp}}$$

Steady-state error for ramp *r(t)*

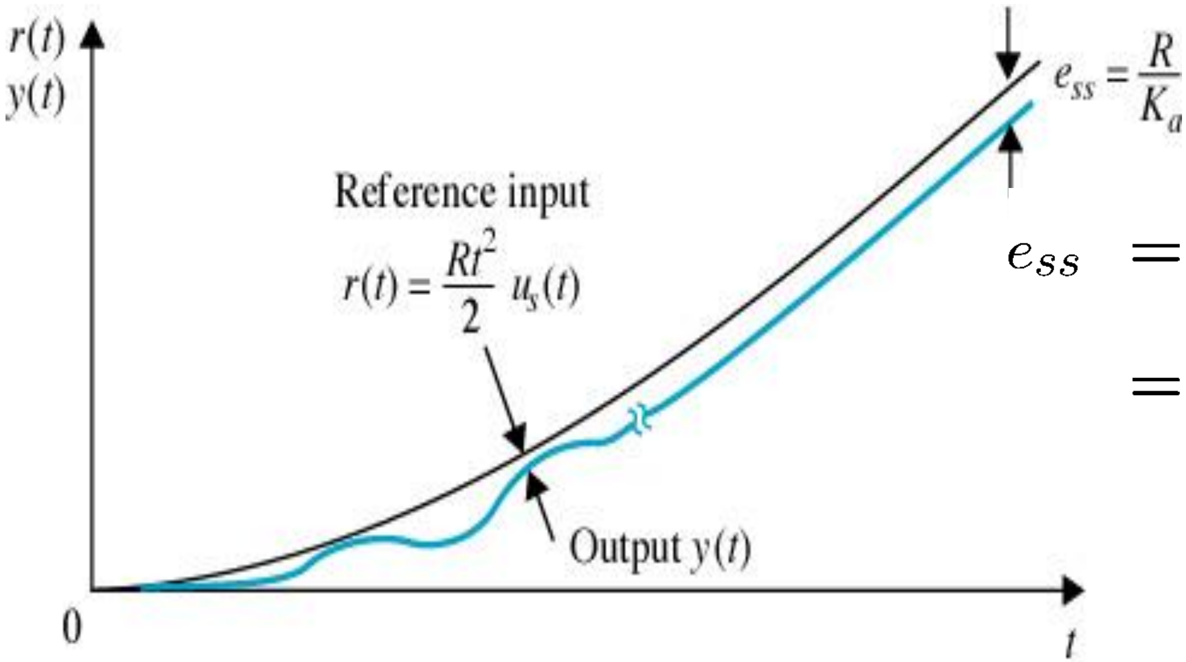
$$r(t) = Rtu(t) \Rightarrow e_{ss} = \frac{R}{K_v}$$



$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s \frac{1}{1+L(s)} \cdot \frac{R}{s^2} \\ &= \frac{R}{\underbrace{\lim_{s\to 0} sL(s)}_{Kv}} \end{aligned}$$

Steady-state error for parabolic *r(t)*

$$r(t) = \frac{Rt^2}{2}u(t) \Rightarrow e_{ss} = \frac{R}{K_a}$$



$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+L(s)} \cdot \frac{R}{s^3}$$
$$= \frac{R}{\underbrace{\lim_{s\to 0} s^2 L(s)}_{Ka}}$$

Zero steady-state error

- When does steady-state error become zero? (i.e. accurate tracking!)
- Infinite error constant!
  - For step *r(t)* $$K_p = \lim_{s \to 0} L(s) = \infty$$

➡ *L(s)* must have at least 1-integrator. (system type 1)

  - For ramp *r(t)* $$K_v = \lim_{s \to 0} sL(s) = \infty$$

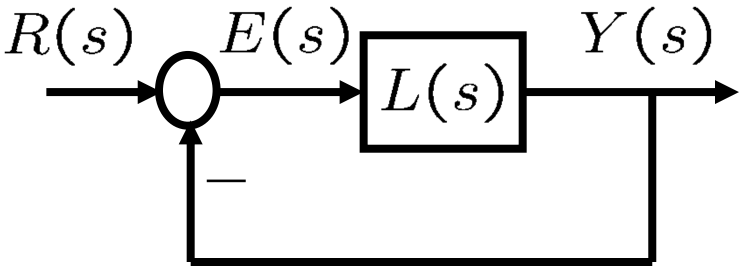
➡ *L(s)* must have at least 2-integrators. (system type 2)

  - For parabolic *r(t)* $$K_a = \lim_{s \to 0} s^2 L(s) = \infty$$

➡ *L(s)* must have at least 3-integrators. (system type 3)

Example 1

- *L(s)* has 2-integrators.



$$L(s) = \frac{K}{s^2(s+12)}$$

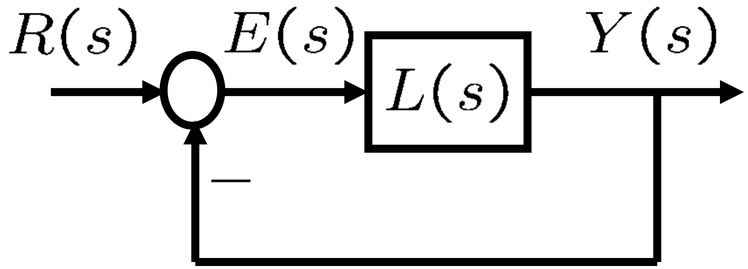
- Characteristic equation

$$1 + L(s) = 0 \Leftrightarrow s^2(s+12) + K = 0 \Leftrightarrow s^3 + 12s^2 + K = 0$$

- CL system is NOT stable for any *K*.
- *e(t)* will not converge. (Don't use today's results if CL system is not stable!!!)

Example 2

- *L(s)* has 1-integrator.



$$L(s) = \frac{K(s+3.15)}{s(s+1.5)(s+0.5)}$$

- By Routh-Hurwitz criterion, CL is stable if

$$0 < K < 1.304$$

- Step *r(t)* $$e_{ss} = \frac{R}{1+K_p} = 0$$

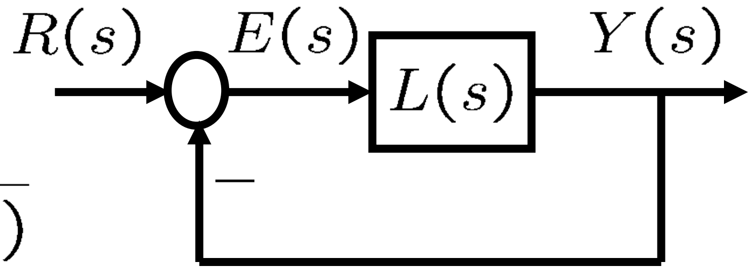
- Ramp *r(t)* $$e_{ss} = \frac{R}{K_v} \quad K_v := \lim_{s \to 0} sL(s) = \frac{3.15K}{0.75} = 4.2K$$

- Parabolic *r(t)* $$e_{ss} = \frac{R}{K_a} = \infty \qquad K_a := \lim_{s \to 0} s^2 L(s) = 0$$

Example 3

- *L(s)* has 2-integrators.

$$L(s) = \frac{5(s+1)}{s^2(s+12)(s+5)}$$



- By Routh-Hurwitz criterion, we can show that CL system is stable.
- Step *r(t)* $$e_{ss} = \frac{R}{1+K_p} = 0$$
- Ramp *r(t)* $$e_{ss} = \frac{R}{K_v} = 0$$
- Parabolic *r(t)* $$e_{ss} = \frac{R}{K_a} = 12R \quad K_a := \lim_{s\to 0} s^2 L(s) = \frac{1}{12}$$

Integrators in *L(s)*

- Integrators in *L(s)* (i.e. plant and controller) are very powerful to eliminate the steady-state errors.
  - Examples 2 & 3
  - Lab 5 – addition of an integral compensator
- However, integrators in *L(s)* tend to destabilize the feedback system.
  - Example 1

Unity Gain Feedback (H(s)=1)

Table 3.1:

System Type Number, N	**Step input:** $\mathbf{r}(t) = Au(t)$	**Ramp input:** $\mathbf{r}(t) = At$	**Parabolic input:** $\mathbf{r}(t) = \frac{A}{2}t^2$
0	$e_{ss} = \dfrac{A}{1+K_p}$	$e_{ss} = \infty$	$e_{ss} = \infty$
1	$e_{ss} = 0$	$e_{ss} = \dfrac{A}{K_v}$	$e_{ss} = \infty$
2	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = \dfrac{A}{K_a}$
≥ 3	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = 0$

Unity Gain Feedback (H(s)=1)

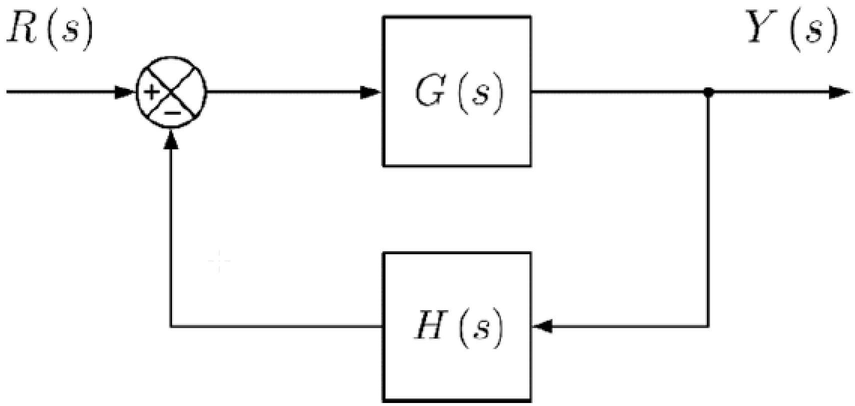
Procedure to determine steady state error:

Given G(s) (and H=1) and input type:

1. Determine the system type. To do this determine the number of poles at zero (i.e. at $s = 0$) of transfer function $G(s)$. Alternatively determine the number of zeros appearing at $s = 0$ of the transfer function: $1 - M(s)$, where $M(s) = \frac{G(s)}{1+G(s)}$, is the closed loop gain.

2. With the system and input types, the steady state error, e_{ss}, can be read from Table 3.1 for most combinations or determined using the appropriate error constant.

Non-Unity Gain Feedback (H(s)≠1)



$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$= \underbrace{\frac{1}{H(s)}}_{\text{Ideal Gain}} \cdot \underbrace{\frac{G(s)H(s)}{1 + G(s)H(s)}}_{\text{Correction term}}$$

Ideal Gain *Correction term*

$$\approx \frac{1}{H(s)}, \quad \text{for } |G(s)H(s)| \gg 1$$

Non-Unity Gain Feedback (H(s)≠1)

DC gain of feedback block: $$k_H \triangleq \lim_{s\to 0} H(s) = H(0)$$

Define steady state error: $$e_{ss} \triangleq \lim_{t\to\infty} \left\{ \frac{1}{k_H} r(t) - y(t) \right\}$$

Final value theorem: $$e_{ss} = \lim_{s\to 0} s \left\{ \frac{1}{k_H} R(s) - Y(s) \right\}$$

Closed loop gain:

$$M(s) = \frac{Y(s)}{R(s)} \implies e_{ss} = \lim_{s\to 0} s \left\{ \frac{1}{k_H} R(s) - M(s)R(s) \right\}$$

$$= \frac{1}{k_H} \lim_{s\to 0} s \left\{ 1 - k_H M(s) \right\} R(s)$$

Non-Unity Gain Feedback (H(s)≠1)

Closed loop gain:

$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Closed loop gain, general form:

$$M(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

Steady state error:

$$e_{ss} = \frac{1}{k_H} \lim_{s\to 0} s\left\{1 - k_H M(s)\right\} R(s)$$

$$\text{System Type} = \#\ \text{zeros at } s = 0 \text{ of } 1 - k_H M(s)$$

Non-Unity Gain Feedback (H(s)≠1)

Table 3.2:

System Type: N	Step input: $\mathbf{r}(t) = Au(t)$	Ramp input: $\mathbf{r}(t) = At$	Parabolic input: $\mathbf{r}(t) = \frac{A}{2}t^2$
0	$e_{ss} = \frac{A}{k_H}\left[\frac{(a_0 - b_0 k_H)}{a_0}\right]$	$e_{ss} = \infty$	$e_{ss} = \infty$
1	$e_{ss} = 0$	$e_{ss} = \frac{A}{k_H}\left[\frac{(a_1 - b_1 k_H)}{a_0}\right]$	$e_{ss} = \infty$
2	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = \frac{A}{k_H}\left[\frac{(a_2 - b_2 k_H)}{a_0}\right]$
≥ 3	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = 0$

$$k_H \triangleq \lim_{s \to 0} H(s) = H(0)$$

Non-Unity Gain Feedback (H(s)≠1)

Procedure to determine steady state error:

Given G(s) and H(s) and input type:

1. Determine constant k_H, the DC gain of $H(s)$, i.e. $k_H \triangleq \lim_{s \to 0} H(s) = H(0)$.

2. Determine the system type. To do this determine the number of zeros appearing at $s = 0$ of the transfer function: $1 - k_H M(s)$, where $M(s) = \frac{G(s)}{1+G(s)H(s)}$, is the closed loop gain.

3. With the system type number and input type, the steady state error, e_{ss}, can be read from Table 3.2 for most combinations or determined using the appropriate coefficients of certain numerator and denominator terms of $M(s)$.

Example 4

$$G(s) = \frac{1}{s} \text{ and } H(s) = \frac{2}{s+4}$$

a) Find the steady state error, e_{ss}, for unit step, ramp and parabolic inputs.

Solution:

Steps:

1) As $H(s) = \frac{2}{s+4}$, the constant $k_H \triangleq \lim_{s\to 0} H(s) = H(0) = \frac{2}{4} = 0.5$

2) The closed loop gain $M(s) = \frac{G(s)}{1+G(s)H(s)}$ so that

$$M(s) = \frac{\frac{1}{s}}{1 + \frac{1}{s} \cdot \frac{2}{s+4}}$$
$$= \frac{s+4}{s^2+4s+2}$$

Example 4, cont'd

$$M(s) = \frac{s+4}{s^2+4s+2}$$

$$\Longrightarrow$$

$$1 - k_H M(s) = 1 - 0.5 \cdot \frac{s+4}{s^2+4s+2}$$
$$= \frac{s(s+3.5)}{s^2+4s+2}$$

One zero at $s = 0 \implies$ system type $= 1$

Example 4, cont'd

From Table 3.2:

System Type: N	Step input: $\mathbf{r}(t) = Au(t)$	Ramp input: $\mathbf{r}(t) = At$	Parabolic input: $\mathbf{r}(t) = \frac{A}{2}t^2$
1	$e_{ss} = 0$	$e_{ss} = \frac{A}{k_H}\left[\frac{(a_1 - b_1 k_H)}{a_0}\right]$	$e_{ss} = \infty$

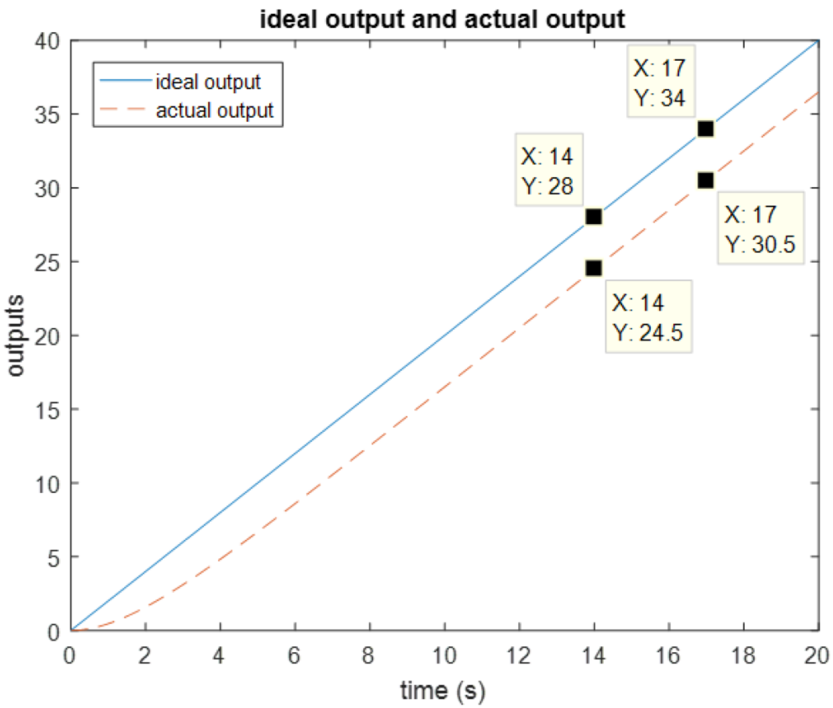
3) With a system type of 1, from the second row of Table 3.2 we see that $e_{ss} = 0$ and $e_{ss} = \infty$ for step and parabolic inputs, respectively. For a unit ramp (with $A = 1$), the error is given by

$$e_{ss} = \frac{1}{k_H}\left[\frac{(a_1 - b_1 k_H)}{a_0}\right]$$

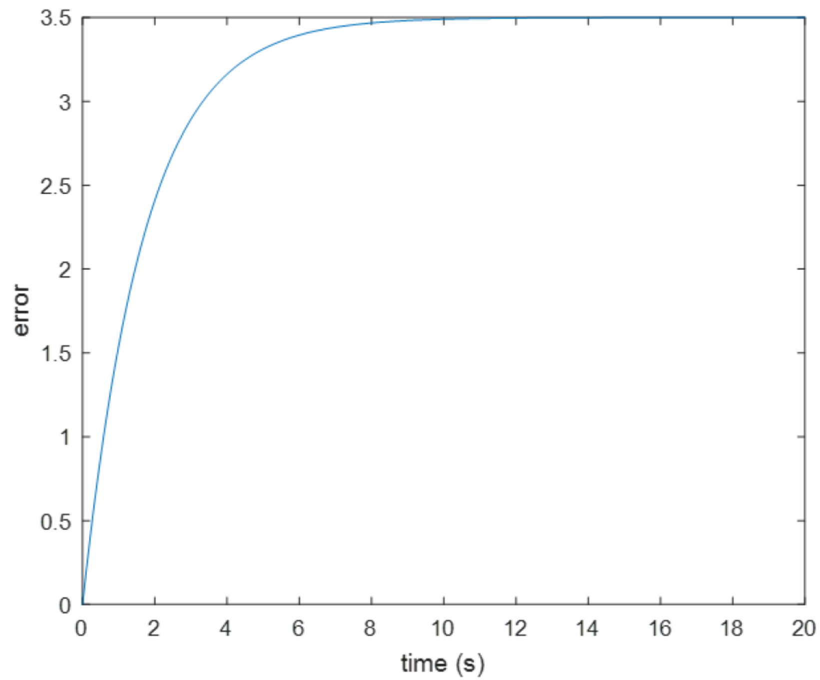
$$= \frac{1}{0.5}\left[\frac{4 - 1 \cdot 0.5}{2}\right] = 3.5$$

Example 4, cont'd

Ideal output and actual output:



Error evolution:



Example 5

$$G(s) = \frac{1}{s^2(s+10)} \text{ and } H(s) = \frac{8(s+1)}{s+4}$$

a) Find the steady state error, e_{ss}, for unit step, ramp and parabolic inputs.

Solution:

Steps:

1) As $H(s) = \frac{8(s+1)}{s+4}$, the constant $k_H = \lim_{s \to 0} H(s) = H(0) = \frac{8}{4} = 2$

2) The closed loop gain, $M(s) = \frac{G(s)}{1+G(s)H(s)}$, so that

$$M(s) = \frac{\frac{1}{s^2(s+10)}}{1 + \frac{1}{s^2(s+10)} \cdot \frac{8(s+1)}{s+4}}$$

Example 5, cont'd

$$M(s) = \frac{s+4}{s^4 + 14s^3 + 40s^2 + 8s + 8}$$

$$\Longrightarrow$$

$$1 - k_H M(s) = 1 - 2 \cdot \frac{s+4}{s^4 + 14s^3 + 40s^2 + 8s + 8}$$

$$= \frac{s(s^3 + 14s^2 + 40s + 6)}{s^4 + 14s^3 + 40s^2 + 8s + 8}$$

One zero at $s = 0 \Longrightarrow$ system type $= 1$

Example 5, cont'd

From Table 3.2:

System Type: N	Step input: $\mathbf{r}(t) = Au(t)$	Ramp input: $\mathbf{r}(t) = At$	Parabolic input: $\mathbf{r}(t) = \frac{A}{2}t^2$
1	$e_{ss} = 0$	$e_{ss} = \frac{A}{k_H}\left[\frac{(a_1 - b_1 k_H)}{a_0}\right]$	$e_{ss} = \infty$

3) With a system type of 1, from the second row of Table 3.2 we see that $e_{ss} = 0$ and $e_{ss} = \infty$ for step and parabolic inputs, respectively. For a unit ramp (with $A = 1$), the error is given by

$$e_{ss} = \frac{1}{k_H}\left[\frac{(a_1 - b_1 k_H)}{a_0}\right]$$

$$= \frac{1}{2}\left[\frac{8 - 1 \cdot 2}{8}\right] = 0.375$$

Example 6: H≠1 Method

(H≠1 method can also be used for H=1 problems)

$$G(s) = \frac{4(s+1)}{s^2(s+10)(s+4)} \text{ and } H(s) = 1$$

a) Find the steady state error, e_{ss}, for unit step, ramp and parabolic inputs.

Solution - Non-Unity Gain Feedback method:

Steps:

1) As $H(s) = 1$, the constant $k_H \triangleq \lim_{s\to 0} H(s) = H(0) = 1$
2) The closed loop gain $M(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{G(s)}{1+G(s)}$ so that

Example 6: H≠1 method, cont'd

$$M(s) = \frac{G}{1+G} = \frac{\frac{4(s+1)}{s^2(s+10)(s+4)}}{1+\frac{4(s+1)}{s^2(s+10)(s+4)}}$$

$$= \frac{4s+4}{s^4+14s^3+40s^2+4s+4}$$

$$\Longrightarrow$$

$$1 - k_H M(s) = 1 - 1 \cdot \frac{4s+4}{s^4+14s^3+40s^2+4s+4}$$

$$= \frac{s^2(s^2+14s+40)}{s^4+14s^3+40s^2+4s+4}$$

Two zeros at $s = 0 \implies$ system type $= 2$

Example 6: H≠1 method, cont'd

From Table 3.2:

System Type: N	Step input: $\mathbf{r}(t) = Au(t)$	Ramp input: $\mathbf{r}(t) = At$	Parabolic input: $\mathbf{r}(t) = \frac{A}{2}t^2$
2	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = \frac{A}{k_H}\left[\frac{(a_2 - b_2 k_H)}{a_0}\right]$

3) With a system type of 2, from the second row of Table 3.2 we see that $e_{ss} = 0$ and $e_{ss} = 0$ for both step and ramp inputs, respectively. For a unit parabola (with $A = 1$), the error is given by

$$e_{ss} = \frac{1}{k_H}\left[\frac{(a_2 - b_2 k_H)}{a_0}\right]$$

$$= \frac{1}{1}\left[\frac{40 - 0 \cdot 1}{4}\right] = 10$$

Example 6: H=1 method

Solution - Unity Gain Feedback method:

$$G(s) = \frac{4(s+1)}{s^2(s+10)(s+4)} \text{ and } H(s) = 1$$

$$\text{Number of poles at } s = 0 \text{ of } G(s) \implies \text{system type} = 2$$

From Table 3.1:

System Type Number, N	**Step input:** $\mathbf{r}(t) = Au(t)$	**Ramp input:** $\mathbf{r}(t) = At$	**Parabolic input:** $\mathbf{r}(t) = \frac{A}{2}t^2$
2	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = \frac{A}{K_a}$

Example 6: H=1 method, cont'd

$$K_a = \lim_{s\to 0} s^2 \cdot G(s)$$

$$= \lim_{s\to 0} s^2 \cdot \frac{4(s+1)}{s^2(s+10)(s+4)}$$

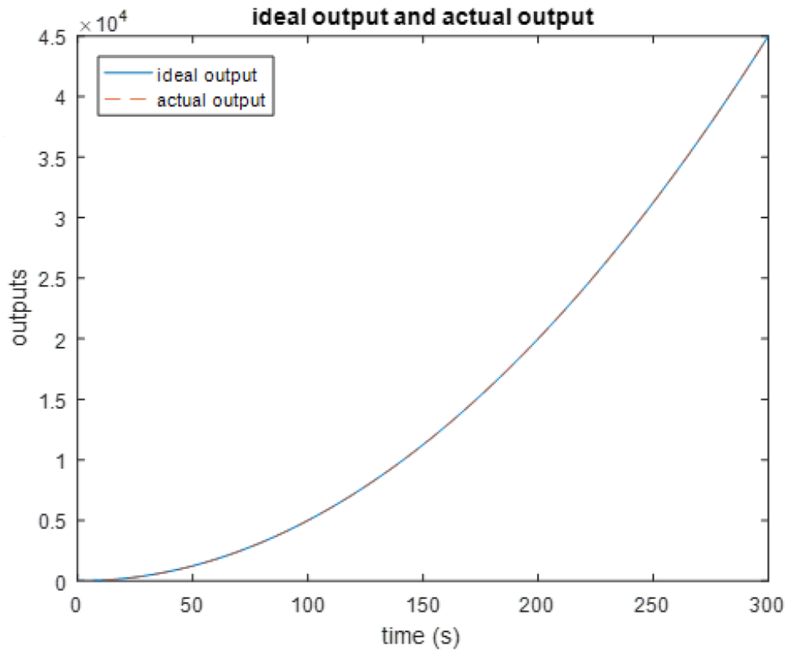
$$= \frac{1}{10}$$

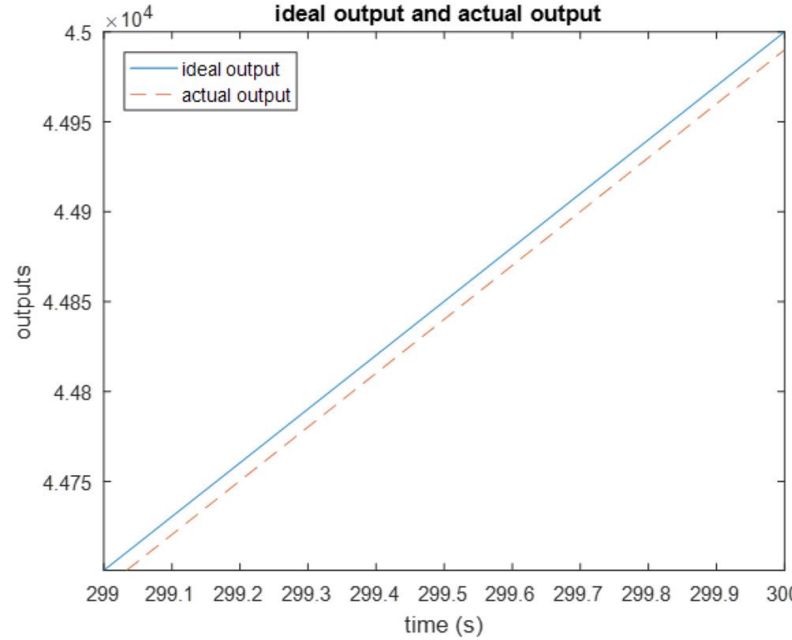
$$e_{ss} = \frac{1}{K_a} \qquad (A=1)$$

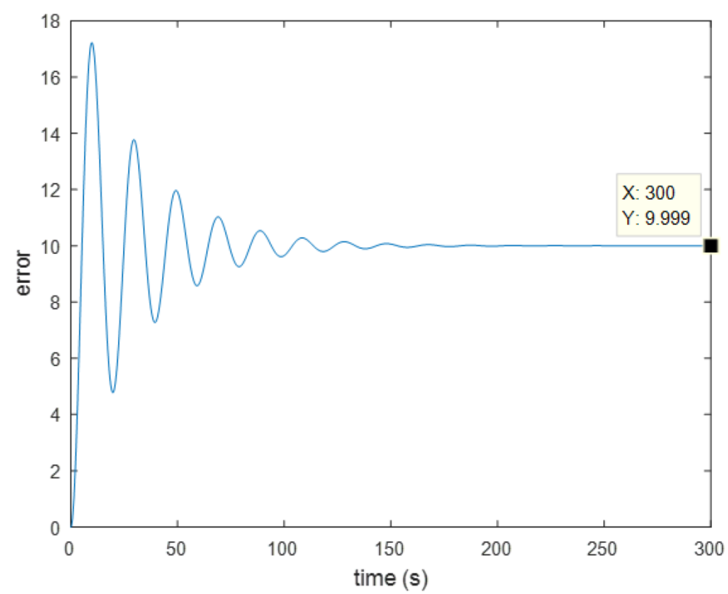
$$= 10$$

Which is the same result obtained by the H≠1 method.

Example 6, cont'd, time response







Summary

- Steady-state error
  - Stable (!) unity and non-unity gain feedback systems are treated in a consistent manner.
  - The key tool is the final value theorem.
  - Main determinants of steady-state error is:
    1) system type
    2) input type